12-2016

Regular Round Matroids

Svetlana Borissova

California State University - San Bernardino, borissos@coyote.csusb.edu

Regular Round Matroids

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Svetlana Borissova

December 2016

Regular Round Matroids

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

by

Svetlana Borissova

December 2016

Approved by:

Dr. Jeremy Aikin, Committee Chair	Date
Dr. Rolland Trapp, Committee Member	
Dr. Belisario Ventura, Committee Member	
Dr. Charles Stanton, Chair, Department of Mathematics	Dr. Corey Dunn Graduate Coordinator, Department of Mathematics

Abstract

A matroid M is a finite set E, called the ground set of M, together with a notion of what it means for subsets of E to be independent. Some matroids, called regular matroids, have the property that all elements in their ground set can be represented by vectors over any field. A matroid is called round if its dual has no two disjoint minimal dependent sets. Roundness is an important property that was very useful in the recent proof of Rota's conjecture, which remained an unsolved problem for 40 years in matroid theory. In this thesis, we give a characterization of round regular matroids.

Acknowledgements

I would like to express my sincere gratitude to my thesis advisor, Dr. Jeremy Aikin, for showing me continuous support, generously sharing his knowledge and professional experience, teaching me to set my goals high and being an example of a true mathematician.

I also want to thank my thesis committee members, Dr. Rolland Trapp and Dr. Belisario Ventura, for their expertise and encouragement.

There are many more people in CSUSB math department who also helped me on my journey. I would like to express my gratitude to the department chair, Dr. Charles Stanton; MA program coordinator, Dr. Corey Dunn, whose classes were the most positive and interesting; my great professors, Dr. Zahid Hasan, Dr. Yuichiro Kakihara, Dr. Giovanna Llosent, Dr. Min-Lin Lo, Dr. Shawn McMurran, Dr. Chetan Prakash and Dr. J. Paul Vicknair, who never ceased to amaze me with their knowledge and passion for math; department administrators, Leeanne Richardson, Allison Torres and Ana Sanchez, for always being friendly and supportive.

Finally, I would like to thank my family for believing in me and showing me support since the time when earning master's degree in mathematics was just a wild idea.

Contents

List of Figures

Chapter 1

Introduction

Matroids were first introduced by the American mathematician Hassler Whitney in his paper "On the abstract properties of linear dependence" published in American Journal of Mathematics in 1935 [Whi35]. In that paper Whitney defined a "system" obeying the following two properties of linear dependence in a matrix:

(a) Any subset of an independent set is independent.

(b) If N_p and N_{p+1} are independent sets of p and $p+1$ columns respectively, then N_p with some column of N_{p+1} forms an independent set of $p+1$ columns.

These two properties not always describe a matrix, so Whitney named any system obeying these properties a "matroid". Moreover, Whitney emphasized a close connection between matroids and graphs. As matroids were studied further, it has been recognized that matroids also combine ideas from combinatorics, finite geometry and abstract algebra. The fact that matroids provide many connections between the various branches of mathematics has been attracting a lot of mathematicians and has made matroid theory one of the most active research areas in mathematics today.

The modern definition of a matroid is given in terms of three independent set axioms:

Definition 1.1 (Matroid)**.** A *matroid* M is defined by a finite set E, called the *ground set*, and a collection $\mathcal{I}$ of subsets of E that satisfy the following axioms:

(I1) $\emptyset \in \mathcal{I}$; (Non-triviality)

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$; (Closed under subsets)

(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$. (Augmentation)

Any subset of E that belongs to $\mathcal{I}$ is called an *independent set*. Subsets of E that are not independent are called *dependent*.

Figure 1.1 illustrates how the same matroid-dependence structure can be represented by the objects from different areas of mathematics.

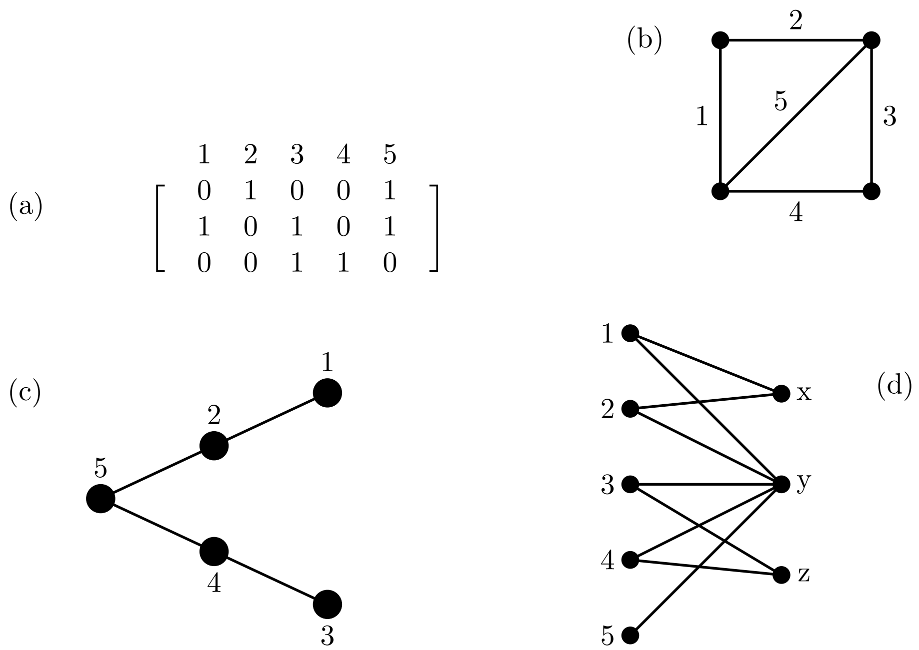


Figure 1.1: Different representations of the same matroid-dependence structure.

All these objects represent the same matoroid $M(E, \mathcal{I})$ with ground set $E = \{1, 2, 3, 4, 5\}$ and collection $\mathcal{I}$ of independent subsets of E consisting of the empty set, all single elements, all pairs of elements and all triples of elements, except $\{1, 2, 5\}$ and $\{3, 4, 5\}$. Clearly, the notion of independence between the elements in each object is different. In matrix (a), independent set corresponds to any subset of columns that are linearly independent. A matroid whose ground set is a set of vectors is called a *representable* matroid. In graph (b), independent set corresponds to an acyclic subset of edges in a graph. A matroid whose ground set is a a set of edges of a graph is called a *cycle* matroid. We will study representable and cycle matroids more closely in Chapter 2.

Picture (c) is a *geometric representation* of matroid $M(E,\mathcal{I})$. It is based on a point-line incidence geometry, so each element of the ground set is represented by a point and every subset of points of size 1 or 2 and any set of points that does not contain a 3-point line is considered independent. In a matroid, an independent set consisting of four elements is represented geometrically by four non-coplanar points. Geometric representation of matroids is discussed in great detail by Gary Gordon and Jennifer McNulty in their book "Matroids: A Geometric Introduction" [GM11].

Picture (d) reflects a bipartite graph. A close connection between matroids and matchings in bipartite graphs was discovered by the German-born British mathematician Richard Rado in the 1940s. A *bipartite* graph is a graph where the set of vertices can be partitioned into two sets so that no edges of the graph join two vertices in the same part of the partition. A subset of edges is called a *matching* if no two edges in the set share a vertex. The collection of all the possible matchings gives us the connection to matroids. In fact, an independent set I of a matroid corresponds to the subset X of the vertices in the same part of the partition that can be matched in a bipartite graph, i.e. there is a matching in which every edge has one endpoint in X. A matroid associated with matchings in a bipartite graph is called a *transversal matroid*. Finding matchings in bipartite graphs is a very important and well-studied topic in combinatorics with applications in scheduling problems, which illustrates the diversity and versatility of matroids.

We will now define other important attributes of matroids. All the matroid notation throughout this thesis will follow Oxley [Oxl11].

Definition 1.2 (Rank)**.** Let $M = (E,\mathcal{I})$ be a matroid and let A be a subset of E. Then the *rank* of A, written $r(A)$, is the size of the largest independent subset of A:

$$r(A) := \max_{I \subseteq A} \{|I| : I \in \mathcal{I}\}.$$

The rank function r satisfies the inequality $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$, making r a submodular function.

Definition 1.3 (Basis)**.** If M is a matroid with independent sets $\mathcal{I}$, then B is a *basis* of the matroid M if B is a maximal (with respect to inclusion) independent set:

$$\mathcal{B} = \{B \in \mathcal{I} \,|\, B \subseteq A \in \mathcal{I} \textit{ implies } B = A\}.$$

Definition 1.4 (Circuit)**.** Let M be a matroid. If C is dependent, but every proper subset of C is independent, we call C a *circuit* in the matroid. Thus, C is a minimal dependent set:

$$\mathcal{C} = \{C \subseteq E \,|\, C \notin \mathcal{I} \text{ and if } I \subsetneq C \text{ then } I \in \mathcal{I}\}.$$

Definition 1.5 (Flat)**.** Let E be the ground set of a matroid M. A subset $F \subseteq E$ is a *flat* if $r(F \cup \{x\}) > r(F)$ for any $x \notin F$. In other words, a flat is a subset of E that is rank-maximal. Thus, adding a new element to a flat increases its rank. A flat is also called a *closed set* of M.

Definition 1.6 (Hyperplane)**.** Let E be the ground set of a matroid M. A subset $H \subseteq E$ is a *hyperplane* if H is a flat of M and if $r(H) = r(M) - 1$.

Definition 1.7 (Closure operator)**.** Let M be an arbitrary matroid having ground set E and rank r. Then the *closure operator* of M is the function cl from 2^E into 2^E defined, for all $X \subseteq E$, by

$$cl(X) = \{x \in E : r(X \cup x) = r(X)\}.$$

We call $cl(X)$ the *closure* of X.

Definition 1.8 (Spanning Set)**.** A subset X of $E(M)$ is a *spanning set* of M if $cl(X) = E(M)$. Equivalently, a subset X is a spanning set if it contains a basis.

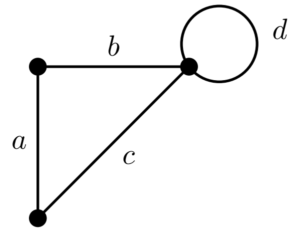
Among other special features of a matroid are loops and coloops. A *coloop* (or isthmus) is an element that is in every basis of the matroid. A *loop* is an element that is in no basis. That is, a loop is a dependent singleton or a circuit of size 1.

Example 1.9. Figure 1.2 shows different representations of a loop d in matroid $M(E, \mathcal{I})$ with ground set $E = \{a, b, c, d\}$ and collection of independent sets $\mathcal{I} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

(a) Matrix representation

$$\begin{array}{cccc} a & b & c & d \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(b) Graph representation



(c) Geometric representation

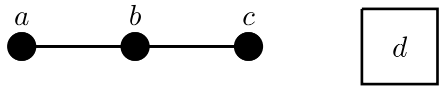
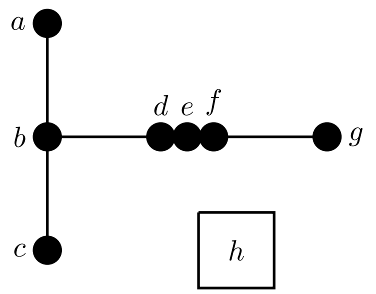


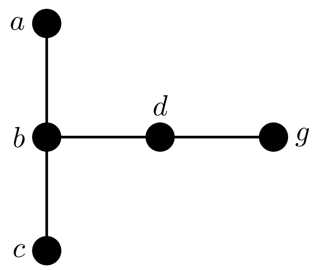
Figure 1.2: Matroid $M(E, \mathcal{I})$ with loop d.

Moreover, if f and g are non-loop elements of a matroid M such that $\{f, g\}$ is a circuit, then f and g are *parallel* in M. A *parallel class* of M is a maximal subset X of $E(M)$ such that any two distinct members of X are parallel and no member of X is a loop. A parallel class is *trivial* if it contains just one element. If we delete all the loops from M and then, in each non-trivial parallel class X, we distinguish one element and delete all the other elements of X, the matroid we obtain is uniquely determined up to renaming of the distinguished elements. We denote this matroid by $si(M)$ and call it the *simplification* of M. If M has no loops and no non-trivial parallel classes, it is called a *simple* matroid.

Example 1.10. Consider matroid M as shown on Figure 1.3(a). This matroid has a non-trivial parallel class $X = \{d, e, f\}$ and a loop h. If we delete loop h and two out of three elements in parallel class X, then we obtain the simplification of M, matroid $si(M)$.



(a) Matroid M



(b) Matroid $si(M)$

Figure 1.3: Simplification of matroid M.

Next we are going to define an isomorphism between two matroids.

Example 1.11. Let G be the graph shown in Figure 1.4 and let matroid $N = N(G)$. Then $E(N) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, \}$ and $\mathcal{C}(N) = \{\{e_8\}\{e_4, e_5\}, \{e_4, e_6\}, \{e_5, e_6\}, \{e_1, e_2, e_3\}, \{e_2, e_4, e_7\}, \{e_2, e_5, e_7\}, \{e_2, e_6, e_7\}\}$. Comparing matroid N with matroid M from Example 1.10, we see that there is a bijection ϕ from $\{a, b, c, d, e, f, g, h\}$ to $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, \}$ defined by:

$$\begin{aligned}
\phi(a) &= e_1\\
\phi(b) &= e_2\\
\phi(c) &= e_3\\
\phi(d) &= e_4\\
\phi(e) &= e_5\\
\phi(f) &= e_6\\
\phi(g) &= e_7\\
\phi(h) &= e_8,
\end{aligned}$$

such that a set C is a circuit in M if and only if $\phi(C)$ is a circuit in N. Equivalently, a set I is independent in M if and only if $\phi(I)$ is independent in N. Thus, the matroids M and N have the same structure and are isomorphic.

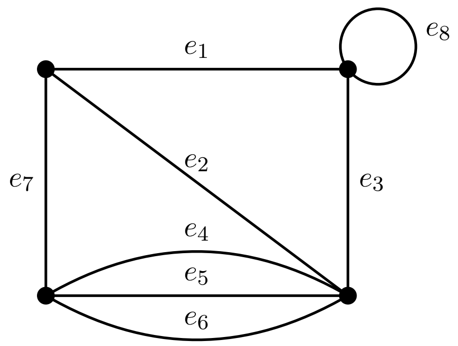


Figure 1.4: Graph G for matroid $N(G)$.

Formally, two matroids M_1 and M_2 are *isomorphic*, written $M_1 \cong M_2$, if there is a bijection ϕ from $E(M_1)$ to $E(M_2)$ such that, for all $X \subset E(M_1)$, the set $\phi(X)$ is independent in M_2 if and only if X is independent in M_1. We call such a bijection ϕ an *isomorphism* from M_1 to M_2.

A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. Therefore, matroid M in Example 1.11 is graphic.

Besides representable, graphic and transversal matroids there is another class of matroids that will be of primary interest in this thesis: *regular* matroids. Regular matroids are a subclass of representable matroids. Their unique feature is that they can be represented by a matrix over any field. Figure 1.5 helps to better understand the relationships between all these classes.

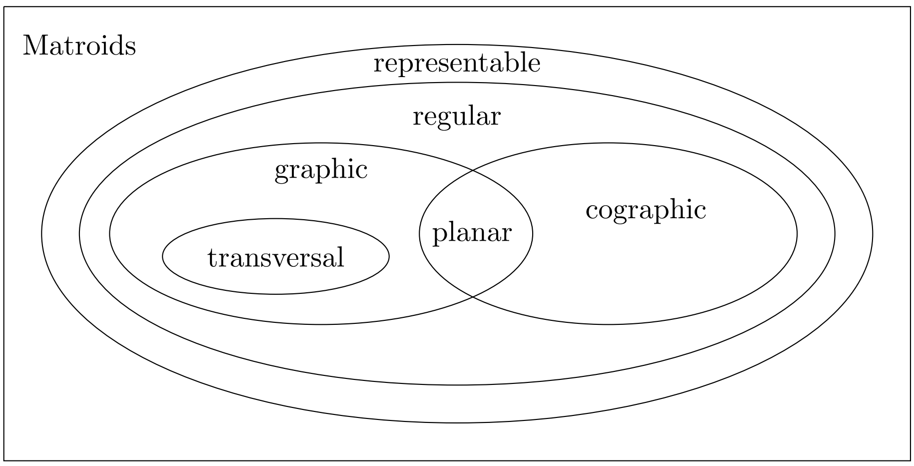


Figure 1.5: Relationships between certain classes of matroids.

Two other classes of matroids in Figure 1.5 that we have not yet mentioned are cographic and planar. To define cographic matroids, we first introduce the notion of duality. Given matroid M on the ground set E, we say that the *dual matroid* M^* is the matroid on the same ground set E, such that $\mathcal{B}(M^*) = \{E - B : B \in \mathcal{B}(M)\}$. The duals of graphic matroids are called *cographic* matroids. If a matroid is both graphic and cographic, then it is isomorphic to the cycle matroid of a planar graph, a graph that can be drawn on a plane without crossing edges. Such matroids are called *planar*.

In this project we will focus on regular matroids that have the additional property of being round. *Round* matroids are an analogue of complete graphs and have the following characterizations:

(i) Matroid M is round if and only if it has no two disjoint cocircuits, where cocircuit is defined by a set $C^* \in E$, such that C^* is a circiut in the dual matroid M^*.

(ii) Matroid M is round if and only if every cocircuit is spanning, i.e. every cocircuit contains a basis.

(iii) Matroid M is round if and only if it cannot be written as the union of two proper flats.

Since regular matroids include graphic and cographic matroids, it would be interesting to know what specific characteristics must be possessed by matroids from these two classes in order to be round. We will answer this question in Section 4.2.

Next we want to see if there are regular round matroids that are neither graphic nor cographic. To investigate this matter we are going to use one of the very important results in matroid theory that was presented by a modern British mathematician, Paul Seymour. In 1980 Paul Seymour published an article titled "Decomposition of Regular Matroids" in which "it is proved that every regular matroid may be constructed by piecing together graphic and cographic matroids and copies of a certain 10-element matroid" [Sey80]. This result won Paul Seymour his first Pólya Prize and is now known as Seymour's Decomposition Theorem stated below.

Theorem 1.12 (Seymour's Decomposition Theorem)**.** *Every regular matroid M can be constructed by using direct sums, 2-sums, and 3-sums starting with matroids each of which* *is either graphic, cographic or isomorphic to R_{10}, and each of which is isomorphic to a minor of M.*

The operations of direct sum, 2-sum and 3-sum allow one to obtain a new matroid from two (or more) arbitrary matroids on disjoint ground sets, ground sets with one element in common and ground sets with a common 3-circuit, respectively. Definitions of these operations along with our results on obtaining a regular round matroids using Seymour's Decomposition Theorem can be found in Chapter 4.

Chapter 2

Classes of Matroids

In this chapter we discuss three important classes of matroids: graphic, cographic and representable matroids.

2.1 Graphic Matroids

There is a close connection between graphs and matroids. To describe this relationship we first need to define graphs.

Definition 2.1 (Graph). *A graph G is a finite nonempty set V of objects called vertices together with a set E of 2-element subsets of V called edges.*

Each edge $\{u, v\}$ of V is commonly denoted by uv or vu. If $e = uv$, then the edge e is said to join vertices u and v and vertices u and v are called the *adjacent vertices*. The number of vertices that are adjacent to a vertex v is called the *degree of* v and denoted by deg v. An edge joining a vertex to itself is called a *loop*. Two or more edges that join the same pair of distinct vertices are called *parallel edges*. A graph G that contains no loops or parallel edges is called a *simple* graph. The number of vertices in a graph G is the *order* of G and the number of edges is the *size* of G.

Two graphs G and G' are *isomorphic* if there exists a bijection $\sigma : V(G) \to v(G')$ such that two vertices u and v are adjacent in G if and only if $\sigma(u)$ and $\sigma(v)$ are adjaent in G'.

Graphs are typically represented by diagrams in which each vertex is represented by a point or small circle (open or solid) and each edge is represented by a line segment or curve joining the corresponding small circles.

Example 2.2. Figure 2.1 shows a graph G with vertex set $V = \{a, b, c, d, e\}$ and edge set $E = \{ab, ac, ad, bb, bc, bd, cd, cd, cd, de\}$. Thus the order of this graph G is 5 and its size is 10. Note that edge bb is a loop and there are three parallel edges joining vertices c and d.

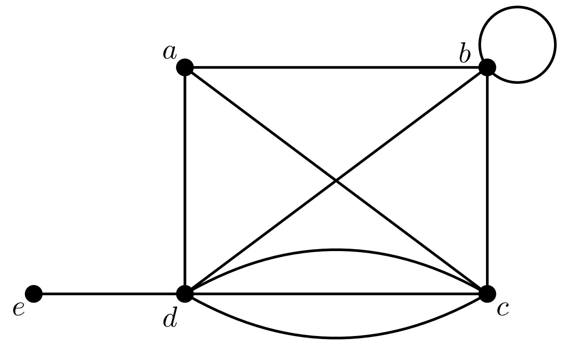


Figure 2.1: A graph G.

If we delete the loop and all except one of parallel edges in graph G, then we obtain a simple graph that we denote $si(G)$ (Figure 2.2).

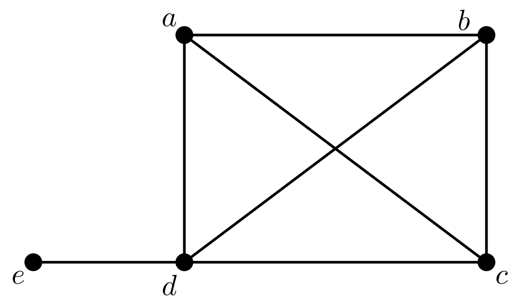


Figure 2.2: A simple graph $si(G)$.

Other important attributes of graphs are defined below.

Definition 2.3 (Walk). *For two (not necessarily distinct) vertices u and v in a graph G, a $u - v$ walk W in G is a sequence of vertices in G, beginning at u and ending at v such that consecutive vertices in W are adjacent in G. A walk whose initial and terminal vertices are distinct is an open walk; otherwise, it is a closed walk. The length of a walk equals the number of edges.*

Definition 2.4 (Path). *A walk in a graph G in which no vertex is repeated is called a path.*

Definition 2.5 (Cycle). *A nontrivial closed walk $C = (v = v_0, v_1, \ldots, v_k = v)$, $k \geq 2$ in which no edge is repeated and the vertices v_i, $1 \leq i \leq k-1$, are distinct is called a cycle. A cycle of length $k \geq 3$ is called a k-cycle. A 3-cycle is also referred to as a triangle.*

Definition 2.6 (Spanning subgraph). *A graph H is said to be a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a spanning subgraph.*

Example 2.7. Figure 2.3 below shows a spanning subgraph of a graph G from Example 2.2. Observe that this subgraph contains a 3-cycle (or triangle) $C = (a, c, d, a)$.

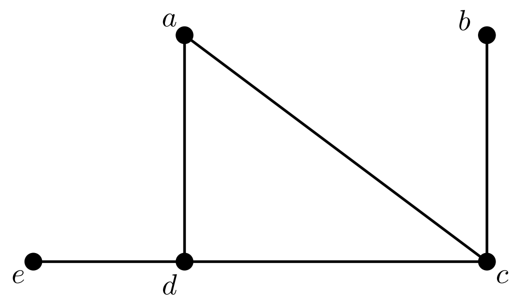


Figure 2.3: Spanning subgraph of graph G from Ex. 2.2.

We can classify graphs in terms of connectivity. We say that graph G is *connected* if for any two vertices u and v, there is a $u-v$ path in G. A graph that is not connected is called *disconnected*. A maximal connected subgraph H of a graph G is called a *component* of G.

There are different degrees of connectedness in graphs. For example, some graphs are so slightly connected that they can be disconnected by the removal of a single vertex or a single edge called *cut-vertex* or a *bridge*, respectively. A nontrivial connected graph that has no cut-vertices is called *nonseparable*. A maximal nonseparable subgraph B of a nontrivial connected graph G is called a *block* of G.

Example 2.8. The graph H in Figure 2.4 is connected since there is a path between every two vertices in H. On the other hand, the graph G is disconnected since, for example, G contains no $y_5 - y_6$ path.

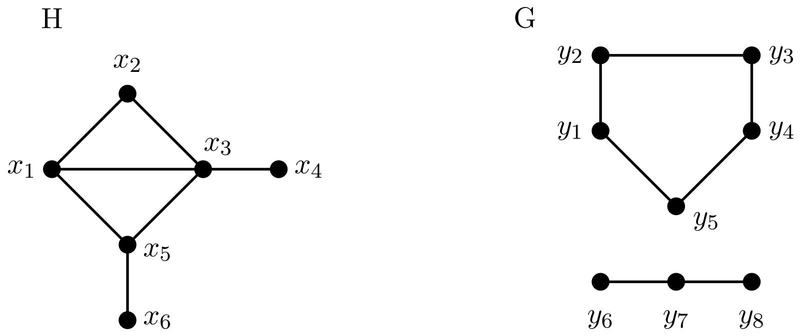


Figure 2.4: A connected graph and disconnected graph.

Three blocks B_1, B_2, B_3 of graph H are shown in Figure 2.5 below.

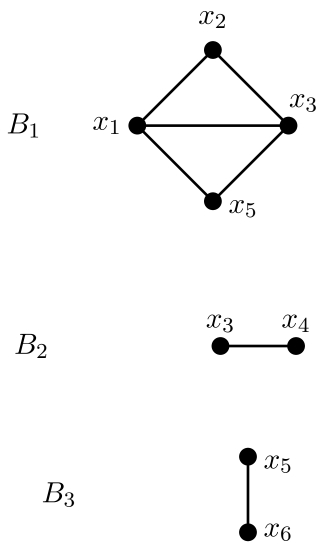


Figure 2.5: The blocks of graph H.

One of the operations that we can perform on a graph is a subdivision. A graph H is a *subdivision* of a graph G if either $H = G$ or H can be obtained from G by inserting vertices of degree 2 into the edges of G. Thus for the graph G in Figure 2.6, graph H is a subdivision of G.

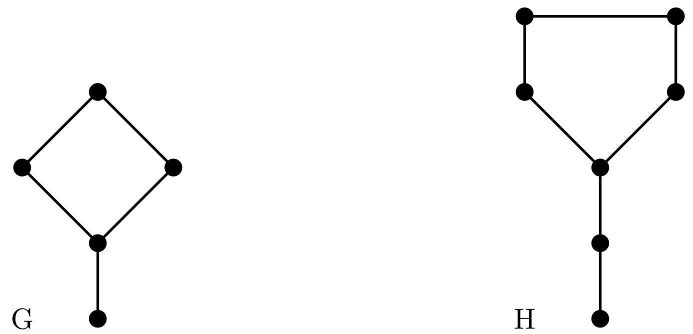


Figure 2.6: Subdivision of graph G.

Among the well-studied classes of graphs are complete and bipartite graphs. A *complete graph* is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge (Figure 2.7). Complete graphs are usually denoted by K_n, where n represents a number of vertices.

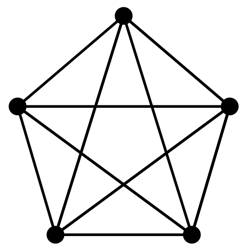


Figure 2.7: The complete graph K_5.

A nontrivial graph G is a *bipartite graph* if it is possible to partition $V(G)$ into two subsets U and W, called partite sets, such that every edge of G joins a vertex of U and a vertex of W. A bipartite graph having partite sets U and W is a *complete bipartite graph* if every vertex of U is adjacent to every vertex of W. If the partite sets U and W of a complete bipartite graph contain s and t vertices, then this graph is denoted by $K_{s,t}$ or $K_{t,s}$. Figure 2.8 shows the complete bipartite graph $K_{3,3}$. Observe that this graph has no disjoint cycles.

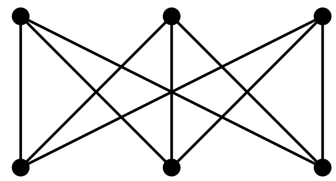
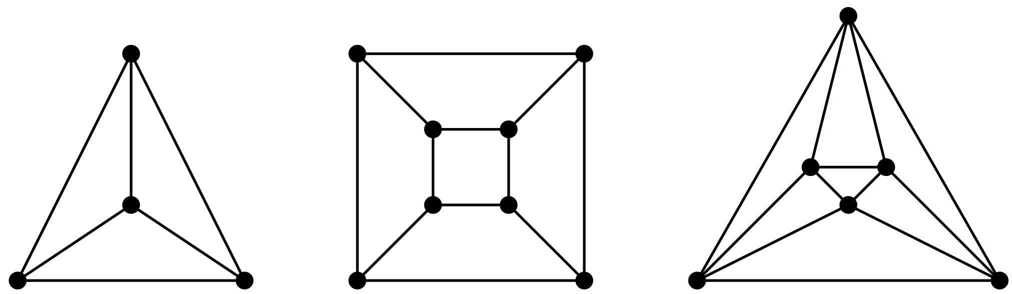


Figure 2.8: The complete bipartite graph $K_{3,3}$.

Another class of graphs is planar graphs. A graph G is called a *planar graph* if G can be drawn in the plane without any two of its edges crossing. Any plane drawing of G divides the plane into regions. Examples of planar graphs include graphs obtained from Platonic solids (Figure 2.9).



Tetrahedron Cube Octahedron

Figure 2.9: Planar graphs.

When considering a plane drawing of graph G of a polyhedron, the faces of the polyhedron become the regions of G, one of which is the exterior region of G.

There exist many interesting results for planar graphs, which can sometimes be used to determine whether a graph is planar or nonplanar. These are some of them:

- For every connected planar graph of order n, size m, and having r regions, $n-m+r=2$ (The Euler Identity).
- If G is a planar graph of order $n \geq 3$ and size m, then $m \leq 3n-6$.
- Every complete graph K_n of order $n \geq 5$ is nonplanar.
- Every planar graph contains a vertex of degree 5 or less.
- The graph $K_{3,3}$ is nonplanar.

Proofs of these properties can be found in most graph theory texts.

Another important result for planar graphs was proved by the well-known Polish topologist Kazimierz Kuratowski in 1930. It provides both necessary and sufficient conditions for a graph to be planar. Kuratowski's Theorem is stated below and will be referred to in the later chapters. A proof of Kuratowski's Theorem can be found in the book titled "Chromatic Graph Theory" by Gary Chartrand and Ping Zhang [CZ09].

Theorem 2.9 (Kuratowski's Theorem)**.** *A graph G is planar if and only of G contains no subgraph that is a subdivision of K_5 or $K_{3,3}$.*

We are now ready to connect graphs to matroids. For any given graph G there is a matroid $M(G)$ associated with it, such that the ground set E corresponds to the set of edges of G and collection of independent sets $\mathcal{I}$ corresponds to the collection of all subsets of edges that are acyclic. Thus, the circuits of such matroid are precisely the cycles of the graph. The matroid $M(G)$ is called the *cycle matroid* of G.

Example 2.10. Consider the graph on the left in Figure 2.10. Its cycle matroid is shown on the right.

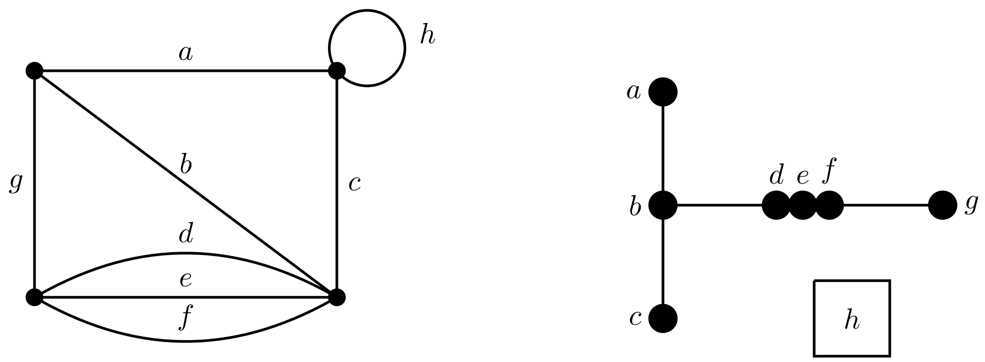


Figure 2.10: A graph and its cycle matroid.

Even though every graph corresponds to a matroid, not every matroid comes from some graph. Matroids that do arise as cycle matroids of graphs are called *graphic*.

The following lemmas provide two properties of graphic matroids.

Lemma 2.11. *Let G and H be graphs. Then $M(G) \cong M(H)$ if and only if $G \cong H$.*

Proof. Let G and H be graphs and $M(G)$ and $M(H)$ be graphic matroids associated with these graphs. Suppose that $G \cong H$. Then there exists a bijection $\sigma : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\sigma(u)$ and $\sigma(v)$ are adjacent in H. Therefore, there is a bijection $\tilde{\sigma} : E(M(G)) \to E(M(H))$ such that, for all $X \subset E(M(G))$, the set $\tilde{\sigma}(X)$ is independent in $M(H)$ if and only if X is independent in $M(G)$. Thus, $M(G) \cong M(H)$.

Now, suppose that $M(G) \cong M(H)$. Then there is a bijection $\tilde{\sigma} : E(M(G)) \to E(M(H))$ such that, for all $X \subset E(M(G))$, the set $\tilde{\sigma}(X)$ is independent in $M(H)$ if and only if X is independent in $M(G)$. Therefore, there exists a bijection $\sigma : V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if $\sigma(u)$ and $\sigma(v)$ are adjacent in H. Thus, $G \cong H$. $\square$

Lemma 2.12. *Let G be a graph. Then $M(si(G)) \cong si(M(G))$, where M is the cycle matroid of G.*

Proof. Let G be a graph. Then the graph $H = si(G)$ is obtained by deleting all the loops and all but one edge in each parallel class in G. Next consider cycle matroid $M(G)$. Then matroid $K = si(M(G))$ is obtained by deleting all the loops and all but one element in each non-trivial parallel class in M. Since $M(G)$ is a cycle matroid, then loops in matroid M correspond to the loops in graph G. Moreover, each non-trivial parallel class in matroid M corresponds to a set of parallel edges in graph G. Therefore, $K \cong M(H)$ and $M(si(G)) \cong si(M(G))$. $\square$

2.2 Cographic Matroids

The definition of cographic matroids is based on the notion of duality. Given matroid M on the ground set E, we say that the *dual matroid* M^* is the matroid on the same ground set E, such that $\mathcal{B}(M^*) = \{E - B : B \in \mathcal{B}(M)\}$. We will study duality in more detail in Chapter 3.

Definition 2.13 (Cographic Matroid)**.** *The dual of a graphic matroid is called a cographic matroid.*

One of the properties of cographic matroids inherited from graph theory is that a graphic matroid is cographic if and only if the corresponding graph is planar.

One more property of cographic matroids is outlined in Lemma 2.14 below.

Lemma 2.14. *Let M be a matroid. If $si(M)$ is cographic, then M is cographic.*

Proof. Let matroid $si(M)$ be the simplification of matroid M. Therefore, $si(M)$ is obtained by deleting all the loops and all but one element in each parallel class in matroid M. Suppose that $si(M)$ is cographic. Then there exists a dual matroid $(si(M))^*$ that is a cycle matroid of some graph G. To restore matroid M from matroid $si(M)$ we would need to add all the deleted loops and elements from parallel classes to matroid $si(M)$. This corresponds to subdividing edges and adding leaves in graph G. Since this new graph corresponds to the dual matroid M^*, then matroid M is cographic. □

2.3 Representable Matroids

2.3.1 Basic Definitions and Examples

The fundamental class of representable matroids is directly connected with matrices and their properties. In fact, any finite set of vectors produces a matroid.

Lemma 2.15. *Let E be the set of column labels of an $m \times n$ matrix A over a field $\mathbb{F}$, and let $\mathcal{I}$ be the set of subsets of X of E for which the multiset of columns labeled by X is a set and is linearly independent in the vector space $V(m, \mathbb{F})$. Then $(E, \mathcal{I})$ is a matroid.*

The matroid obtained from the matrix A is called the *vector matroid* of A and denoted by $M[A]$. Moreover, any matroid M that is isomorphic to the vector matroid of a matrix D over a field $\mathbb{F}$ is representable over $\mathbb{F}$ or $\mathbb{F}$-representable, and D is a representation for M over $\mathbb{F}$ or an $\mathbb{F}$-representation for M. A matroid that is representable over some field is called *representable.*

We already stated that ground set E of representable matroid $M[A]$ corresponds to the set of columns of matrix A and collection of independent sets $\mathcal{I}$ corresponds to the linearly independent sets of columns of matrix A. Below is a list of other important attributes of representable matroid and their correlation with the matrix:

- Bases ($\mathcal{B}$) correspond to the maximal linearly independent sets of columns.
- Circuits ($\mathcal{C}$) correspond to the minimal linearly dependent sets of columns.
- Rank (r) corresponds to the rank of the matrix.
- Flats ($\mathcal{F}$) correspond to the sets of columns equal to their linear span.

- Hyperplanes ($\mathcal{H}$) correspond to flats of rank one less than the rank of the matrix.
- Closure correspond to the linear span.
- Spanning sets ($\mathcal{S}$) correspond to the subsets of columns whose linear span contains all the columns of the matrix.

Next example illustrates these relations.

Example 2.16. Consider matrix A in Figure 2.9 represented over the field $\mathbb{R}$ of real numbers.

$$\begin{array}{c} \\ \text{A=} \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 2.11: Matrix A.

Then the vector matroid $M[A]$ on ground set $E = \{1, 2, 3, 4, 5\}$ has the following attributes:

- Rank $r(M[A]) = 3$ is the rank of matrix A.
- Collection of independent sets $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{1, 2, 3\}\}$ represent the collection of subsets of linearly independent columns of matrix A.
- Bases $\mathcal{B} = \{\{1, 2, 3\}, \{1, 3, 5\}, \{2, 3, 5\}\}$ is the maximal set of columns that are linearly independent.
- Collection of circuits $\mathcal{C} = \{\{1, 2, 3\}, \{4\}\}$ corresponds to the minimal linearly dependent sets of columns of matrix A. In general, these are harder to recognizance in a matrix.
- Subcollection of hyperplanes of $M[A]$ that can be found using the above representation of matrix A is $\{\{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4, 5\}, \}$. Recall that hyperplane is viewed as a set of columns of rank 1 less than the rank of a matrix and that is equal to its linear span. If we fix any non-zero row in matrix A, then the set of columns with zero entries in that row forms a subspace of the rank equal to $rank(A) - 1$ and thus represents a hyperplane.

In general, $M[A]$ does not uniquely determine A. A vector matroid M remains unchanged if one performs any of the following elementary row operations on A:

(i) Interchange two rows.

(ii) Multiply a row by a non-zero member of $\mathbb{F}$.

(iii) Replace a row by the sum of that row and another.

(iv) Adjoin or remove a zero row.

(v) Interchange two columns (the labels moving with the columns).

(vi) Multiply a column by a non-zero member of $\mathbb{F}$.

(vii) Replace each matrix entry by its image under some automorphism of $\mathbb{F}$.

Assume that the matrix A is non-zero. Using the above elementary row operations, we can reduce A to the form $[I_r|D]$ where I_r is the $r \times r$ identity matrix and D is some $r \times (n-r)$ matrix over $\mathbb{F}$. Clearly, $r = r(M)$. If we label the columns of $[I_r|D]$ by $e_1, e_2, ..., e_n$, then $\{e_1, e_2, ..., e_r\}$ is a basis B of M. Moreover, it is natural to label the rows of D, in order, by $e_1, e_2, ..., e_r$. Thus $M[A]$ can be represented both by the matrix $[I_r|D]$, whose columns are labeled $e_1, e_2, ..., e_n$ (Figure 2.10), and by the matrix D, whose row are labeled $e_1, e_2, ..., e_r$ and whose columns are labeled $e_{r+1}, e_{r+2}, ..., e_n$ (Figure 2.11). Matrix $[I_r|D]$ is called a *standard representation matrix* for M and matrix D is called a *reduced standard representative matrix.*

$$\begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & \dots & e_r & e_{r+1} & e_{r+2} & \dots & e_n \end{array} \\ \left[\begin{array}{c|c} \qquad I_r \qquad & \qquad D \qquad \end{array}\right] \end{array}$$

Figure 2.12: Standard representative matrix for M.

$$\begin{array}{c} \\ e_1 \\ e_2 \\ \vdots \\ e_{r-1} \\ e_r \end{array} \begin{array}{c} \begin{array}{cccc} e_{r+1} & e_{r+2} & \cdots & e_n \end{array} \\ \left[\begin{array}{ccc} & & \\ & & \\ & D & \\ & & \\ & & \end{array} \right] \end{array}$$

Figure 2.13: Reduced standard representative matrix for M.

Besides the standard representative matrix, a vector matroid M can be represented by the *circuit incidence matrix*. If M is a matroid on the set $\{1, 2, \dots, n\}$ such that $\mathcal{C}(M) = \{C_1, C_2, \dots, C_m\}$, then the circuit incidence matrix $A(\mathcal{C})$ of M is the $m \times n$ matrix $[a_{ij}]$ in which a_{ij} is 1 or 0 depending on whether j is or is not in C_i.

Example 2.17. Consider matroid M from Example 1.10. It has ground set $E(M) = \{a, b, c, d, e, f, g, h\}$ and the set of circuits $\mathcal{C}(M) = \{C_1, C_2, \dots, C_8\}$, where $C_1 = \{h\}, C_2 = \{d, e\}, C_3 = \{d, f\}, C_4 = \{e, f\}, C_5 = \{a, b, c\}, C_6 = \{b, d, g\}, C_7 = \{b, e, g\}$ and $C_8 = \{b, f, g\}$. The circuit incidence matrix $A(\mathcal{C})$ of M is the 8×8 matrix shown below.

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Figure 2.14: The circuit incidence matrix of matroid M from Ex.1.11.

The most commonly studied classes of representable matroids are binary, ternary and regular matroids. Binary and ternary matroids are representable over $GF(2)$ and $GF(3)$ respectively. A regular matroid is one that can be represented over any field.

2.3.2 Binary Matroids

Binary matroids are representable over $GF(2)$ and have a number of special properties that are not possessed by matroids in general. For that reason, binary matroids have been widely-studied and characterized.

One of the unique properties of binary matroids connects the cocircuit space of a binary matroid to the row space of a matrix by which it's represented. In general, the row space $\mathcal{R}(A)$ of an $m \times n$ matrix A over a field $\mathbb{F}$ is the subspace of $V(n, \mathbb{F})$ that is spanned by the rows of A. This property is outlined in Lemma 2.16.

Lemma 2.18. *Let A be a binary representation of a rank-r binary matroid M. Then the cocircuit space of M equals the row space of A. Moreover, this space has dimension r and is the orthogonal subspace of the circuit space of M.*

Example 2.19. Consider matroid M represented by the binary matrix A and graph G in Figure 2.13 (a) and (b) respectively.

(a)

$$A= \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(b)

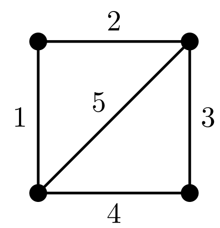


Figure 2.15: Matrix (a) and graph (b) representations of matroid M.

The members of the row space $\mathcal{R}(A)$ of matrix A are the rows of the following matrix.

$$\begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 1 + Row 2} \\ \text{Row 1 + Row 3} \\ \text{Row 2 + Row 3} \\ \text{Row 1 + Row 2 + Row 3} \\ \text{Row 1 + Row 1} \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Figure 2.16: Row space $\mathcal{R}(A)$ of matrix A.

Viewed as incidence vectors, the rows of this matrix correspond to the sets

$\{\{2,5\},\{1,3,5\},\{3,4\},\{1,2,3\},\{2,3,4,5\},\{1,4,5\},\{1,2,4\},\{\emptyset\},\}$ in matroid M. By finding these sets on graph G we can check that they represent all possible disjoint unions of cocircuits.

2.3.3 Regular Matroids

The following statements are equivalent for a matroid M:

(i) M is regular.

(ii) M is representable over every field.

(iii) M is binary and, for some field $\mathbb{F}$ of characteristic other than two, M is $\mathbb{F}$-representable.

Sometimes regular matroids are referred to as *unimodular matroids*, because they can be represented by a totally unimodular matrix. A *totally unimodular matrix* is a matrix over $\mathbb{R}$ for which every square submatrix has determinant in the set $\{-1,0,1\}$. Such matrices play an important role in computer science in solving liner programming problems.

It's also important to note that every graphic matroid and every cographic matroid is regular.

One of the well-studied representatives of the class of regular matroids is matroid R_{10} that is the vector matroid of the matrix A_{10} over $GF(2)$ shown on Figure 2.17.

$$
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
A_{10} = \left[\begin{array}{c|ccccc}
 & 1 & 1 & 0 & 0 & 1 \\
 & 1 & 1 & 1 & 0 & 0 \\
I_5 & 0 & 1 & 1 & 1 & 0 \\
 & 0 & 0 & 1 & 1 & 1 \\
 & 1 & 0 & 0 & 1 & 1
\end{array}\right]
\end{array}
$$

Figure 2.17: Matrix representation for regular matroid R_{10}.

The matroid R_{10} has many attractive features. These are some of them:

- Among regular matoids that are neither graphic nor cographic, the only one with ten elements and the only simple one of rank at most five.
- The dual of matroid R_{10} is isomorphic to R_{10}.

- Every single-element deletion is isomorphic to $M(K_{3,3})$, and every single-element contraction is isomorphic to $M^*(K_{3,3})$.
- Every circuit has four or six elements.

Chapter 3

Matroid Constructions

In this chapter we will introduce several different ways of obtaining a new matroid from one or more arbitrary matroids.

3.1 Duality

One of the most important properties of matroids is duality.

Definition 3.1 (Dual Matroid)**.** Let M be a matroid on the ground set E. Then the dual matroid M^* is a matroid on the same ground set E, so that

$$\mathcal{B}(M^*) = \{E - B : B \in \mathcal{B}(M)\}.$$

Matroid M is called *self-dual* if $M \cong M^*$ and *identically self-dual* if $M = M^*$. The bases, independent sets, spanning sets, circuits and hyperplanes of the dual matroid M^* are related to those of M as follows:

M^*		M
B is a basis	$\Leftrightarrow$	$E - B$ is a basis
I is independent	$\Leftrightarrow$	$E - I$ is spanning
S is spanning	$\Leftrightarrow$	$E - S$ is independent
C is a circuit	$\Leftrightarrow$	$E - C$ is a hyperplane
H is a hyperplane	$\Leftrightarrow$	$E - H$ is a circuit

Evidently, $r(M) + r(M^*) = |E(M)|$.

Another important attribute of a matroid that we can define using duality is a *cocircuit.* A cocircuit of a matroid $M(E, \mathcal{I})$ is a set $C^* \subseteq E$, such that C^* is a circuit in the dual matroid M^*. Equivalently, we can look at cocircuits as hyperplane complements. In a cycle matroid $M(G)$, each cocircuit corresponds to a minimal edge cut-set of G, which is a collection of edges whose removal from the graph breaks a component into two or more pieces. For example, if we take all the edges in a connected graph G that are incident to a given vertex (that is not a cut-vertex), then we get a cocircuit in $M(G)$. That cocircuit is the complement of a hyperplane in $M(G)$, because adding another edge to that hyperplane gives a spanning set.

Example 3.2. Consider graph G shown in Figure 3.1. The vertex cut-set $\{d, g, j, h\}$ disconnects vertex v_4 from the graph. Therefore, $\{d, g, j, h\}$ is a cocircuit in the cycle matroid $M(G)$. Observe, that the cocircuit $\{d, g, j, h\}$ is the compliment of the hyperplane $H = \{a, b, c, e, f, i, k, l\}$, because adding one of the edges d, g, k or h to H will give us a spanning set.

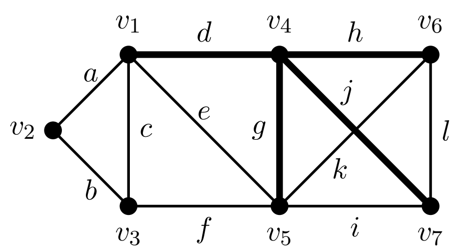


Figure 3.1: Graph G.

For any representable matroid $M[A]$, cocircuits, viewed as hyperplane compliments, correspond to the subset of columns with non-zero entries in any fixed row of matrix A. Note that any given representation of a matrix does not provide the entire collection of cocircuits in a matroid, since performing row operations on the matrix A will allow us to see more cocircuits, when viewed from this perspective.

One of the properties of duality is outlined in the next result.

Lemma 3.3. *If a matroid M is representable over a field $\mathbb{F}$, then M^* is also representable over $\mathbb{F}$.*

In particular, the dual of a binary matroid is binary, and the dual of a ternary matroid is ternary.

3.2 Minors

Deletion and contraction are two important operations that we can perform on a matroid. Both operations reduce the size of the matroid by removing an element from $E(M)$.

Definition 3.4 (Deletion)**.** *Let M be a matroid on the ground set E with independent sets $\mathcal{I}$. For $e \in E$ (e is not a coloop), the matroid $M \backslash e$ has ground set $E - \{e\}$ and independent sets that are those members of $\mathcal{I}$ that do not contain e. In other words, I is independent in $M \backslash e$ if and only if $e \notin I$ and I is independent in M.*

Definition 3.5 (Contraction)**.** *Let M be a matroid on the ground set E with independent sets $\mathcal{I}$. For $e \in E$ (e is not a loop), the matroid M/e has ground set $E - \{e\}$ and independent sets that are formed by choosing all those members of $\mathcal{I}$ that contain e, and then removing e from each set. In other words, $I - \{e\}$ is independent in M/e if and only if $e \in I$ and I is independent in M.*

Combining and iterating these operations produces a *minor* of the original matroid.

3.3 Series and Parallel Connection

The operations of joining electrical components in series and in parallel are fundamental in electrical network theory. There also exist the corresponding operations for graphs that naturally extend to matroids. First, we are going to investigate these operations for graphs.

Example 3.6. Consider graphs G, G_1 and G_2 as shown in Figure 3.2. Graphs G_1 and G_2 were obtained from graph G by adding the edge f *in parallel* with edge e and *in series* with e, respectively. We call G_1 a parallel extension of G and G_2 a series extension of G.

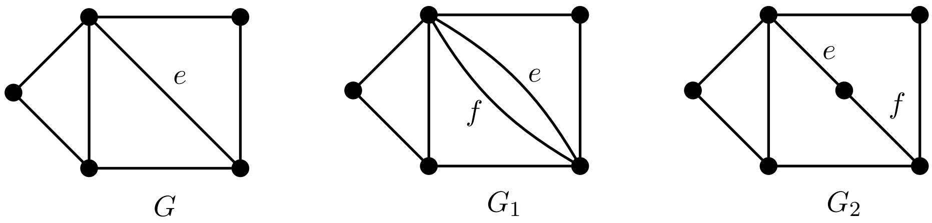


Figure 3.2: Parallel extensions and series extensions.

For an arbitrary graph G, these operations are defined as follows: G' is a *parallel extension* of G, or, equivalently, G is a *parallel deletion* of G' if G' has a two-edge cycle $\{e, f\}$ such that $G' \backslash f = G$. If, instead, $\{e, f\}$ is a two-edge cocycle of G', and $G'/f = G$, then G' is a *series extension* of G, and G is a *series contraction* of G'. Note, that not every series extension consists of replacing an edge by a path of length two.

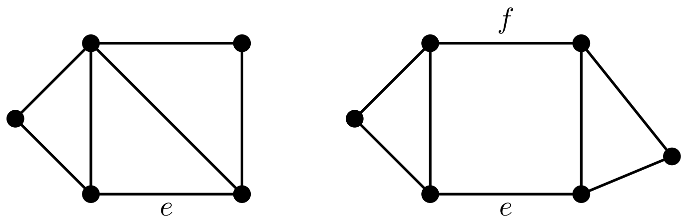


Figure 3.3: A series extension that is not a subdivision.

The operations of series and parallel extensions in graphs can be generalized to matroids. In particular, if $M \backslash f = N$ and f is in a 2-circuit of M, then M is a parallel extension of N, and N is a parallel deletion of M. If, instead, $M/f = N$ and f is in a 2-cocircuit of M, then M is a *series extension* of N, and N is a series contraction of M. Clearly M is a parallel extension of N if and only if M^* is a series extension of N^*. A series class of M is a parallel class of M^*; it is non-trivial if it has at least two elements. Moreover, a *series minor* of a matroid M is a matroid N that is obtained from M by a series of deletions and series contractions. If, instead, N can be obtained from M by a sequence of contractions and parallel deletions, then N is a parallel minor of M. Clearly, M_1 is a parallel minor of M_2 if and only if M_1^* is a series minor of M_2^*.

The operations of series and parallel extension for graphs are special cases of the operations of *series and parallel connection* of graphs. We will define these operations and show how they naturally extend to matroids. For each i in $\{1, 2\}$, let p_i be an edge of a graph G_i. Arbitrarily assign a direction to p_i and label its tail by u_i and its head by

v_i. To form the series and parallel connections of G_1 and G_2 with respect to the direct edges p_1 and p_2, we begin by deleting p_1 from G_1 and p_2 from G_2; we then identify u_1 and u_2 as the vertex u. To complete the series connection, we add a new edge p joining v_1 and v_2. The parallel connection is completed by identifying v_1 and v_2 as the vertex v and then adding a new edge p joining u and v. Thus, unless exactly one of p_1 and p_2 is a loop, the parallel connection is obtained by simply identifying p_1 and p_2 so that their directions agree.

Example 3.7. The graphs G and H in Figure 3.4 are, respectively, the series and parallel connections of the graphs G_1 and G_2 with respect to the directed edges p_1 and p_2.

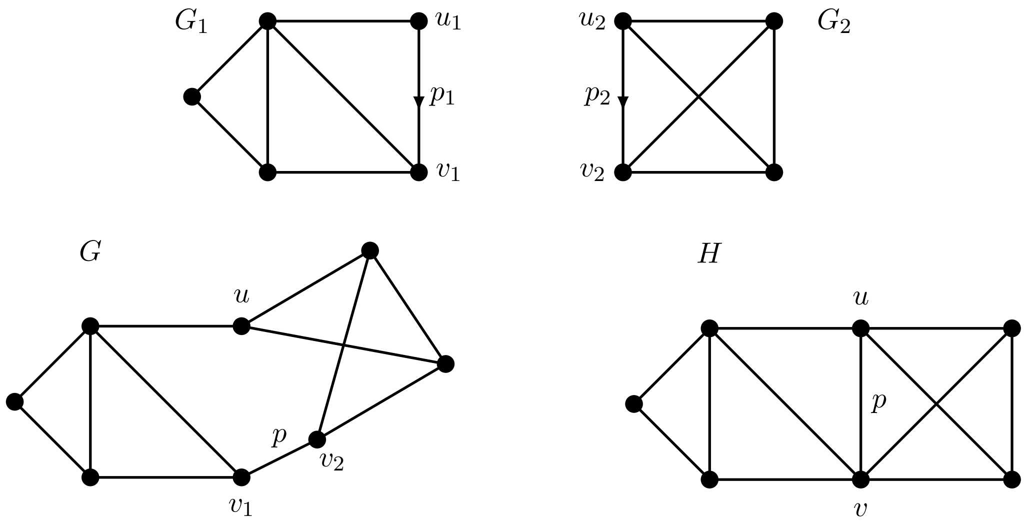


Figure 3.4: Series and parallel connection in graphs.

Now we will show how series and parallel connection in graphs can be extended to matoroids. Let $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\mathcal{P}}$ denote the collection of circuits of the cycle matroids of the series and parallel connections of the graphs G_1 and G_2. Then in the last example, and indeed in general, it is not difficult to specify $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\mathcal{P}}$ in terms of $\mathcal{C}(M(G_1))$ and $\mathcal{C}(M(G_2))$. Writing M_1 for $M(G_1)$ and M_2 for $M(G_2)$ and assuming neither p_1 nor p_2 is a loop or a cut edge, we have

$$\mathcal{C}_{\mathcal{S}} = \mathcal{C}(M_1 \backslash p_1) \cup \mathcal{C}(M_2 \backslash p_2) \cup \{(C_1 - p_1) \cup (C_2 - p_2) \cup p : p_i \in C_i \in \mathcal{C}(M_i) \text{ for each } i\}$$

and

$\mathcal{C}_{\mathcal{P}} = \mathcal{C}(M_1\backslash p_1) \cup \{C_1 - p_1 \cup p : p_1 \in C_1 \in \mathcal{C}(M_1)\} \cup \mathcal{C}(M_2\backslash p_2) \cup \{(C_2 - p_2) \cup p : p_2 \in C_2 \in \mathcal{C}(M_2)\} \cup \{(C_1 - p_1) \cup (C_2 - p_2) : p_i \in C_i \in M_i \text{ for each } i\}$.

Now suppose that M_1 and M_2 are arbitrary matroids on disjoint sets. Let p_1 and p_2 be elements of M_1 and M_2, respectively, such that neither p_1 nor p_2 is a loop or coloop. Take p to be an element that is not in $E(M_1)$ or $E(M_2)$ and let $E = E(M_1\backslash p_1) \cup E(M_2\backslash p_2) \cup p$. Then each of $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\mathcal{P}}$ is the collection of circuits of a matroid on E. These matroids are denoted by $S((M_1; p_1), (M_2; p_2))$ and $P((M_1; p_1), (M_2; p_2))$, or briefly, $S(M_1, M_2)$ and $P(M_1, M_2)$, and called the series and parallel connections of M_1 and M_2 with respect to the basepoints p_1 and p_2.

It is often convenient to view $S(M_1, M_2)$ and $P(M_1, M_2)$ as being formed from two matroids M_1 and M_2 whose ground sets meet in a single element p. In this context, p is called the *basepoint* of the connection and we take $E = E(M_1) \cup E(M_2)$. Moreover, with $p_1 = p_2 = p$, the sets $\mathcal{C}_{\mathcal{S}}$ and $\mathcal{C}_{\mathcal{P}}$ are defined as above provided neither M_1 nor M_2 has p as a loop or a coloop.

Example 3.8. Let both M_1 and M_2 be isomorphic to the uniform matroid $U_{2,4}$ whose ground set E has 4 elements and the collection of independents sets $\mathcal{I}$ includes all subsets of E with 2 or fewer elements. Then geometric representation for $S(M_1, M_2)$ and $P(M_1, M_2)$ are given in Figure 3.5. In matroid $S(M_1, M_2)$, the basepoint p is free in space, that is, p is in no circuits of size less than five, so the rank of matroid $S(M_1, M_2)$ is 4. Matroid $P(M_1, M_2)$ was obtained by "gluing" together M_1 and M_2 at p. Thus, the rank of matroid $P(M_1, M_2)$ is 3.

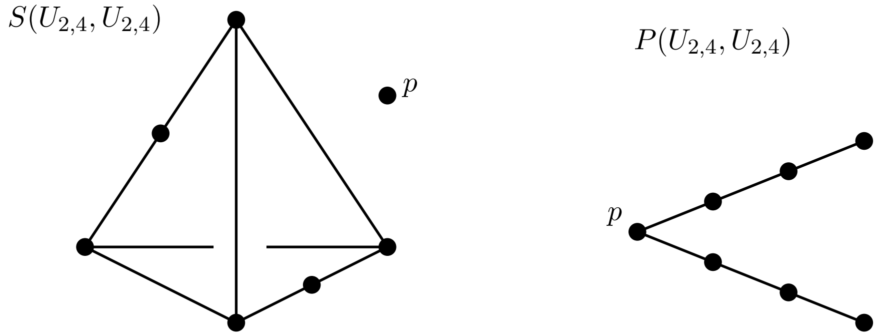


Figure 3.5: Series and parallel connections of two matroids $U_{2,4}$.

In the above example, our observation of the ranks of matroids $S(M_1, M_2)$ and

$P(M_1, M_2)$ can be generalized by the following property:

$$
r(S(M_1, M_2)) = \begin{cases} r(M_1) + r(M_2) - 1, & \textit{if } p \textit{ is a coloop of both } M_1 \text{ and } M_2; \\ r(M_1) + r(M_2) & \textit{otherwise.} \end{cases}
$$

$$
r(P(M_1, M_2)) = \begin{cases} r(M_1) + r(M_2), & \textit{if } p \textit{ is a coloop of both } M_1 \text{ and } M_2; \\ r(M_1) + r(M_2) - 1 & \textit{otherwise.} \end{cases}
$$

Another important property of the operations of series and parallel connection is given in the next lemma.

Lemma 3.9. *Let M_1 and M_2 be matroids with $E(M_1) \cap E(M_2) = \{p\}$. Then $S(M_1, M_2)/p = P(M_1, M_2)\backslash p$.*

We have seen that both the series and parallel connections of two graphic matroids are graphic. We now consider the effect of the operations of series and parallel connection on representable matroids.

Proposition 3.10. *Let $\mathbb{F}$ be a field. If M_1 and M_2 are $\mathbb{F}$-representable matroids such that $E(M_1) \cap E(M_2) = \{p\}$, then both $P(M_1, M_2)$ and $S(M_1, M_2)$ are $\mathbb{F}$-representable.*

The matrix in Figure 3.6 is a totally unimodular representation for $P(M_1, M_2)$.

$$
\begin{array}{c}
\begin{array}{ccc} E(M_1) - p & p & E(M_2) - p \end{array} \\
\left[\begin{array}{c|c|c}
 & 0 & \\
 & 0 & \\
A_1 & \vdots & 0 \\
 & 0 & \\
 & 0 & \\
\hline
 & 1 & \\
\hline
 & 0 & \\
 & 0 & \\
0 & \vdots & A_2 \\
 & 0 & \\
 & 0 &
\end{array}\right]
\end{array}
$$

Figure 3.6: Matrix representation of $P(M_1, M_2)$.

The matrix in Figure 3.7 is a totally unimodular representation for $S(M_1, M_2)$.

$$
\begin{array}{ccc}
E(M_1)-p & p & E(M_2)-p \\
\end{array}
$$
$$
\left[\begin{array}{c|c|c}
 & 0 & \\
 & 0 & \\
A_1 & \vdots & 0 \\
 & 0 & \\
 & 1 & \\
\hline
 & 1 & \\
 & 0 & \\
0 & \vdots & A_2 \\
 & 0 & \\
 & 0 & \\
\end{array}\right]
$$

Figure 3.7: Matrix representation of $S(M_1, M_2)$.

The notion of the operation of parallel connection of two matroids with one common element can be extended and generalized to an operation that joins matroids with more than one common element. We begin by defining some fundamental matroid constructions and their properties.

Definition 3.11 (Restriction). *Let M be the matroid $(E, \mathcal{I})$ and suppose that $X \subseteq E$. Let $\mathcal{I}|X$ be $\{I \subseteq X : I \in \mathcal{I}\}$. Then the pair $(X, \mathcal{I}|X)$ is a matroid. We call this matroid the restriction of M to X.*

Suppose that the matroids M_1 and M_2 have ground sets E_1 and E_2, rank functions r_1 and r_2, and closure operators cl_1 and cl_2. Let $E_1 \cup E_2 = E$. Assume that $M_1|T = M_2|T = N$ where $E_1 \cap E_2 = T$. The rank function of this common restriction of M_1 and M_2 will be denoted by r. If M is a matroid on E such that $M|E_1 = M_1$ and $M|E_2 = M_2$ then M is called an *amalgam* of M_1 and M_2.

If M is an arbitrary amalgam of M_1 and M_2, then by submodularity of the rank function, for all $X \subseteq E$,

$$r_M(X) \leq \eta(X)$$

where

$$\eta(X) = r_1(X \cap E_1) + r_2(X \cap E_2) - r(X \cap T).$$

Now let

$$\zeta(X) = \min\{\eta(Y) : Y \supseteq X\}.$$

Then, for all $X \subseteq E$,

$$\zeta(X) \geq r_M(X).$$

When ζ is submodular, the matroid E that has ζ as its rank function is called the *proper amalgam* of M_1 and M_2. Now, let the simple matroid associated with $M_1|T$, $\mathrm{si}(M_1|T)$, be denoted by $\mathrm{si}(T)$, where $\mathrm{si}(T)$ is a modular flat of $\mathrm{si}(M_1)$. Then the proper amalgam of M_1 and M_2 is called the *generalized parallel connection* of M_1 and M_2 *across* T. This matroid will be denoted by $P_N(M_1, M_2)$ or $P_T(M_1, M_2)$, where we recall that $N = M_1|T = M_2|T$.

Chapter 4

Regular Round Matroids

In this chapter we will focus on regular round matroids. First, we will define round matroids and show what characteristics graphic and cographic matroids must possess in order to be round. Next, we will use Seymour's Decomposition theorem to determine the existence of other regular round matroids that are neither graphic nor cographic.

4.1 Round Matroids

Round matroids are an analogue of complete graphs and defined by the following equivalent statements:

(i) Matroid M is round if and only if it has no two disjoint cocircuits.

(ii) Matroid M is round if and only if every cocircuit is spanning, i.e. every cocircuit contains a basis.

(iii) Matroid M is round if and only if it cannot be written as the union of two proper flats.

Lemma 4.1 below introduces one of the properties of round matroids.

Lemma 4.1. *If $si(M)$ is round, then M is round.*

Proof. Let $M(E, \mathcal{I})$ be a matroid and $si(M)$ be the simplification of M. Let $\{X_1, X_2, ..., X_k\}$ be the set of all parallel classes in M such that $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,m} : x_{i,j} \in E(M)\}$.

Without loss of generality, assume that $x_{1,1}, x_{2,1}, \ldots, x_{k,1} \in E(si(M))$. Since $r(X_i) = 1$ for all i, then whenever $x_{i,j} \in H$, where H is a hyperplane in M, it must be that $X_i \in H$. Moreover, whenever $x_{i,j} \in C^*$, where C^* is a cocircuit in M, then $X_i \in C^*$.

Suppose M is not round. Then there exist disjoint index sets $\mathcal{A}$ and $\mathcal{B}$ such that $\bigcup\limits_{a\in\mathcal{A}} X_a$ and $\bigcup\limits_{b\in\mathcal{B}} X_b$ are disjoint cocircuits in M. Then in $si(M)$, $C_1^* = \{x_{a,1} : a \in \mathcal{A}\}$ and $C_2^* = \{x_{b,1} : b \in \mathcal{B}\}$ are two disjoint cocircuits, a contradiction.

□

Now we will show what characteristics must be possessed by graphic matroids in order to be round.

Theorem 4.2. *A graphic matroid $M(G)$ is round if and only if $si(M(G)) \cong M(K_n)$, for some $n \geq 2$, where K_n is a complete graph of order n.*

Proof. Suppose that graphic matroid $M(G)$ is round. Then every cocircuit in $M(G)$ is spanning. Therefore, every corresponding minimal edge-cut set in graph G is a spanning set. Since the collection of edges incident to any vertex in G forms a minimal edge-cut set, then $si(G)$ must be isomorphic to some complete graph K_n, $n \geq 2$. By Lemmas 2.9 and 2.10, $si(M(G)) \cong M(K_n)$.

Now suppose that $si(M(G)) \cong M(K_n)$, for some $n \geq 2$, and $si(M(G))$ is not round. Then there exists a minimal edge-cut set E' in $si(G)$ that is not spanning. Therefore there exists a vertex v in $si(G)$ such that E' does not contain an edge incident with v. But $si(G) \cong K_n$, therefore all edges adjacent to v form a spanning tree, a contradiction.

□

Theorem 4.4 below shows what characteristics must be possessed by cographic matroids in order to be round. First we prove the following Lemma.

Lemma 4.3. *Let $M(E)$ and N be matroids. If $M \cong N$, then $si(M) \cong si(N)$.*

Proof. Suppose that $M \cong N$. Then there exists a bijection ϕ from $E(M)$ to $E(N)$ such that for all $X \subset E(M)$, the set $\phi(X)$ is independent in N if and only if X is independent in M. Consider the restriction $\phi' : E(si(M)) \to E(si(N))$ of ϕ. Since ϕ is a bijection, then ϕ' is also a bijection. Moreover, since ϕ maps independent sets to independent sets, then ϕ' must map independent sets to independent sets. Therefore, $si(M) \cong si(N)$.

□

Theorem 4.4. *Matroid N is a cographic round matroid if and only if $si(N)$ is either isomorphic to $M(K_n)$ for some $n \leq 4$, or to $M^*(K_{3,3})$.*

Proof. Let $N(E, \mathcal{I})$ be a matroid. Suppose that $si(N)$ is isomorphic to $M(K_n)$ for some $n \leq 4$, or to $M^*(K_{3,3})$. We want to show that N is cographic and round.

If $si(N)$ is isomorphic to $M(K_n)$, $n \leq 4$, then $si(N)$ is cographic, since K_n is planar for $n \leq 4$. Moreover, since $deg(v) = n - 1$ for all $v \in V(K_n)$, then every cocircuit in $M(K_n)$ is spanning. Thus, $si(N) \cong M(K_n)$ is round. Then by Lemmas 2.2 and 4.1, N is cographic and round.

If $si(N)$ is isomorphic to $M^*(K_{3,3})$, then N is cographic, since $(si(N))^* \cong M(K_{3,3})$ is graphic. Also, since $K_{3,3}$ does not have any two disjoint cycles, $M(K_{3,3})$ has no two disjoint circuits. Thus, $M^*(K_{3,3})$ has no two disjoint cocircuits and, therefore, is round. Since $si(N) \cong M^*(K_{3,3})$, then $si(N)$ is round. By Lemma 2.2 and Lemma 4.1, N is cographic and round.

Now suppose that N is cographic and round. We want to show that $si(N)$ is isomorphic to $M(K_n)$, for some $n \leq 4$, or to $M^*(K_{3,3})$.

<u>Case 1.</u> Suppose that N is graphic. Then $N \cong N(G)$ is the cycle matroid of some connected graph $G(V, E)$. Therefore, $N(G)$ is also round, graphic and cographic. Then G must be a planar graph with every minimal edge cut-set being spanning. Suppose $v \in V(G)$, such that v is not a cut-vertex. Since all the edges that are incident with v form a minimal edge cut-set that separates v from the rest of the graph, then v must be adjacent to every vertex in G. Now, suppose that $v \in V(G)$ and v is a cut-vertex. Then the proper subset of edges incident with v form a minimal edge cut-set. Therefore this cut-set can not be spanning, a contradiction. Thus, graph G has no cut-vertices and any two vertices in G are adjacent.

Let $si(G)$ be a graph obtained from G by removing all but one edge from each parallel class and all loops in $E(G)$. Then $si(G) \cong K_n$ is a complete planar graph and there exists a cycle matroid $M(K_n) \cong si(N(G))$ by Lemma 2.1. Since $N(G) \cong N$, then $si(N(G)) \cong si(N)$ by Lemma 1.12. Therefore, $si(N) \cong si(N(G)) \cong M(K_n)$, $n \leq 4$.

<u>Case 2.</u> Suppose that N is not graphic. Since N is cographic, then there exist a nonplanar graph G, such that $P(G)$ is a cycle matroid and $P(G) \cong N^*$.

Since N is round, then N has no two disjoint cocircuits and $P(G) \cong N^*$ has no two disjoint circuits. Thus, G has no two disjoint cycles. Moreover, since G is nonplanar, then by Kuratowskis Theorem, G must contain a subgraph that is a subdivision of K_5 or $K_{3,3}$. But any subdivision of K_5 has at least two disjoint cycles. Therefore, G must be a graph with no two disjoint cycles that contains a subgraph that is a subdivision of $K_{3,3}$.

Let $B_1, B_2, ..., B_n$ be a partition of $E(G)$ such that B_i is a block in G. Since G must contain at least one subgraph that is a subdivision of $K_{3,3}$, then without loss of generality, let B_1 be such a subgraph. Moreover, since G has no two disjoint cycles, then there exists only one block containing all the cycles of G. Thus, B_1 must contain all the cycles of G. Therefore, $B_2, B_3, ..., B_n$ are subgraphs of G consisting of single edges.

Since the union of all subgraphs B_i, $2 \leq i \leq n$, in G forms a forest, then the corresponding subset S_i of elements of matroid P is a set of coloops. Thus, S_i is a set of loops in P^*. Moreover, in B_1 all subdivided edges correspond with series classes in P, which are parallel classes in P^*. Since the simplification of P^* will involve deletion of all the loops and all but one element from parallel classes, it follows that $si(P^*)$ is isomorphic to a cographic matroid $M^*(G')$ such that G' can be obtained from graph G by deleting all the single edge blocks and contracting all the subdivided edges. Therefore, $G' \cong K_{3,3}$ and $si(P^*) \cong M^*(K_{3,3})$. Moreover, since $P \cong N^*$, then $P^* \cong N$. So, by Lemma 1.12, $si(N) \cong si(P*) \cong M^*(K_{3,3})$.

$\square$

We have introduced the class of round matroids as well as the class of regular matroids (see Section 2.3.4) and are now ready to study matroids that are both regular and round. In particular, we want to know what characteristics these matroids possess. Recall that regular matroids include graphic and cographic matroids. Thus, the characteristics of round matroids that are stated in Theorems 4.2 and 4.4 hold for certain regular round matroids. That is:

- If regular matroid M is graphic, then it is round when the corresponding graph G is complete and of order $n \geq 2$ (see Theorem 4.2).
- If regular matroid N is cographic, then it is round when its simplification is either isomorphic to a matroid $M(K_n)$ for some $n \leq 4$, or to $M^*(K_{3,3})$ (see Theorem 4.4).

The next step will be to investigate regular round matroids that do not fall into the two categories mentioned above. To do that, we will use Seymour's Decomposition

Theorem stated in the next section.

4.2 Seymour's Decomposition Theorem

Proved by the British mathematician Paul Seymour in 1980, Seymour's Decomposition Theorem plays an important role in matroid theory. According to this theorem, every regular matroid can be obtained from a number of graphic matroids, cographic matroids and copies of R_{10}. The operations that are used to stick these building blocks together are the direct sum, 2-sum, and 3-sum. These operations allow one to obtain a new matroid from two (or more) arbitrary matroids on disjoint ground sets, ground sets with one element in common, and ground sets with a common 3-circuit, respectively. We introduce these operations in more detail in the next three sections.

Seymour's Decomposition Theorem is formally stated below.

Theorem 4.5. *Every regular matroid M can be constructed by using direct sums, 2-sums, and 3-sums starting with matroids each of which is either graphic, cographic, or isomorphic to R_{10}, and each of which is isomorphic to a minor of M.*

Our goal is to see if we can construct a regular round matroid using this theorem. We are going to examine each operation referenced in the theorem and conclude whether it allows us to obtain a regular round matroid.

4.3 The Operation of Direct Sum

The operation of direct sum allows one to form a new matroid from two or more arbitrary matroids on disjoint sets.

Proposition 4.6. *Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)\}$. Then $(E, \mathcal{I})$ is a matroid.*

The matroid $(E, \mathcal{I})$ in the last proposition is called the *direct sum* or *1-sum* of M_1 and M_2 and denoted by $M_1 \oplus M_2$. Clearly, $M_1 \oplus M_2 = M_2 \oplus M_1$. More generally, for n matroids, $M_1, M_2, \ldots, M_n$, on disjoint sets, $E_1, E_2, \ldots, E_n$, the direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ is the pair $(E, \mathcal{I})$ where $E = E_1 \cup E_2 \cup \cdots \cup E_n$ and $\mathcal{I} = \{I_1 \cup I_2 \cup \cdots \cup I_n : I_i \in \mathcal{I}(M_i)$ for all i in $\{1, 2, \ldots, n\}\}$. In this case, $M_1 \oplus M_2 \oplus \cdots \oplus M_n$

is also a matroid. Matroids $M_1, M_2, \ldots, M_n$ are called the *direct sum components* of $M_1 \oplus M_2 \oplus \cdots \oplus M_n$.

The next proposition provides two basic properties of the direct sum. These are stated for $M_1 \oplus M_2$ but can easily be extended to the direct sum of an arbitrary number of matroids.

Proposition 4.7. *Let M_1, M_2 be matroids defined on disjoint ground sets E_1 and E_2 with independent sets $\mathcal{I}(M_1)$ and $\mathcal{I}(M_2)$, respectively. Then,*

(i) *Bases: $\mathcal{B}(M_1 \oplus M_2) = \{B_1 \cup B_2 : B_1 \in \mathcal{B}(M_1)$ and $B_2 \in \mathcal{B}(M_2)\}$.*

(ii) *Cocircuits: $\mathcal{C}^*(M_1 \oplus M_2) = \mathcal{C}^*(M_1) \cup \mathcal{C}^*(M_2)$.*

Example 4.8. Let M_1 be the four-point line on the ground set $\{1, 2, 3, 4\}$ and let M_2 be the matroid on the ground set $\{5, 6, 7, 8\}$ with circuits $\mathcal{C} = \{\{5, 6, 7\}, \{5, 6, 8\}, \{7, 8\}\}$ (see Figure 4.1). Then an independent set of $M_1 \oplus M_2$ is formed by taking the union of an independent set in M_1 with one from M_2. For example, the set $\{1, 3, 5\}$ is independent in $M_1 \oplus M_2$. Note, that $r(M_1 \oplus M_2) = r(M_1) + r(M_2) = 2 + 2 = 4$, since a basis for $M_1 \oplus M_2$ is simply the union of a basis of M_1 with a basis of M_2.

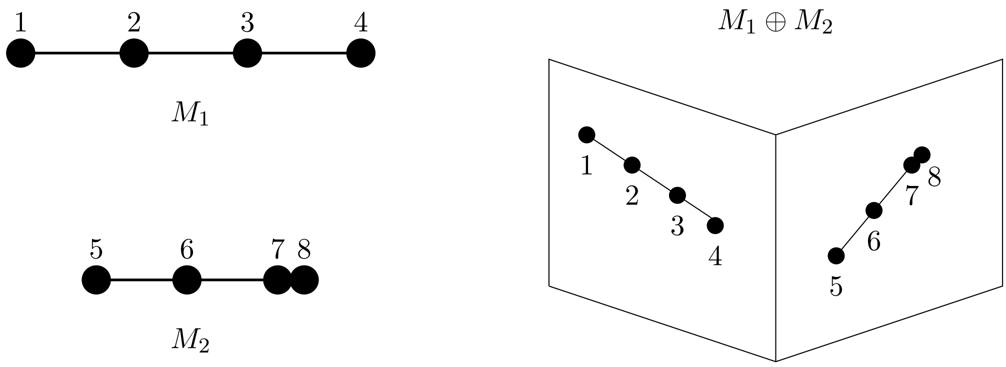


Figure 4.1: The direct sum $M_1 \oplus M_2$.

In our attempt to construct regular round matroids using Seymour's Decomposition Theorem, we are going to consider the direct sum and show that we can not obtain a round matroid by using this operation.

Proposition 4.9. *Let M_1 and M_2 be matroids. Then the direct sum $M_1 \oplus M_2$ is never round.*

Proof. Let M_1 and M_2 be matroids with the sets of cocircuits $\mathcal{C}^*(M_1)$ and $\mathcal{C}^*(M_2)$ respectively. Then by Proposition 4.8(v), $\mathcal{C}^*(M_1 \oplus M_2) = \mathcal{C}^*(M_1) \cup \mathcal{C}^*(M_2)$. Therefore, matroid $M = M_1 \oplus M_2$ has disjoint cocircuits and, thus, is not round. □

In the next section we will investigate the operation of 2-sum.

4.4 The Operation of 2-sum

The operation of 2-sum allows one to join matroids with exactly one common element. Before we define the operation of 2-sum for all matroids, we are going to see how we can apply this operation to graphs and cycle matroids.

Example 4.10. Consider graphs G_1 and G_2 shown on Figure 4.2 (a) and (b). If we assume that $M(G_1)$ and $M(G_2)$ are the corresponding cycle matroids, then the graphs in (c) and (d) are isomorphic to the matroids $P(M(G_1), M(G_2))$ and $S(M(G_1), M(G_2))$, the parallel connection and series connection of $M(G_1)$ and $M(G_2)$ with respect to the basepoint p. Observe, that the graph in (e) can be obtained both from the graph in (c) by contracting p and from the graph in (d) by deleting p. The cycle matroid of this graph is isomorphic to both $P(M(G_1), M(G_2))\backslash p$ and $S(M(G_1), M(G_2))/p$.

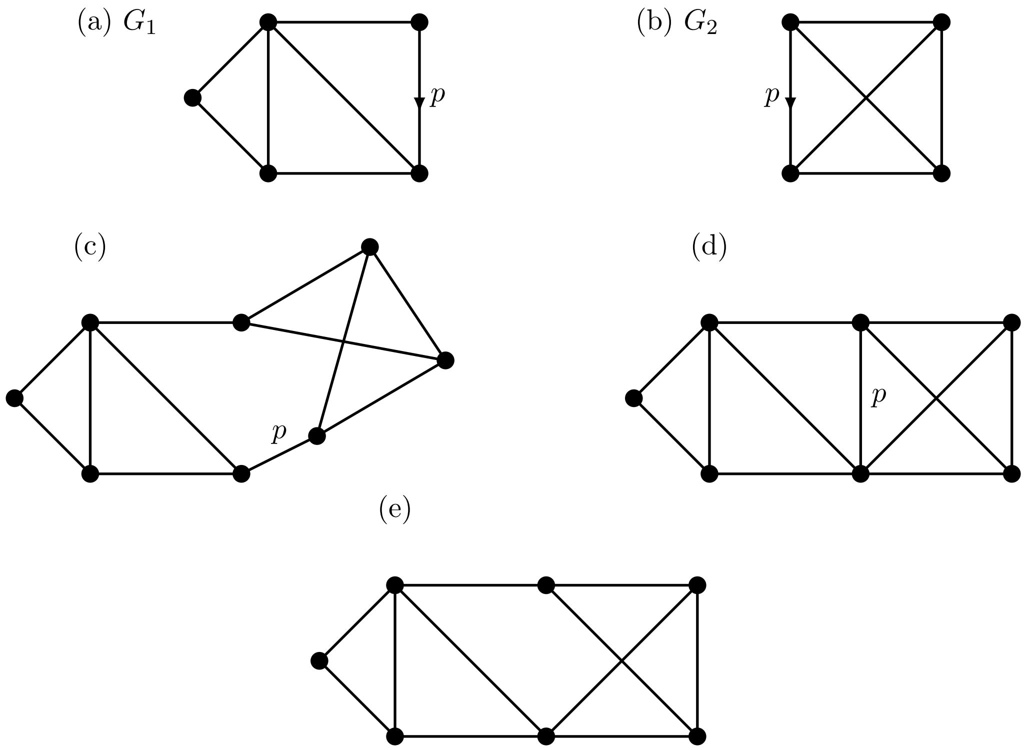


Figure 4.2: The parallel and series connections and 2-sum of graphs.

Moreover, the graph in (e) can also be obtained directly from G_1 and G_2 by identifying the edges labeled p and then deleting the identified edge. We call a graph obtained in this way a 2-sum of G_1 and G_2. To ensure that this operation is well-defined, we insist that if the edge p is a loop in one of G_1 and G_2, then it is a loop in the other.

By analogy with the above operation for graphs, there is the following definition for matroids.

Definition 4.11 (2-sum of matoirds)**.** *Let M and N be matroids, each with at least two elements. Let $E(M) \cap E(N) = \{p\}$ ad suppose that neither M and N has $\{p\}$ as a separator. Then the 2-sum $M \oplus_2 N$ of M and N is $S(M, N)/p$ or, equivalently, $P(M, N)\backslash p$.*

Clearly $M \oplus_2 N = N \oplus_2 M$. The element p is called the *basepoint* of the 2-sum, and M and N are called the *parts* of the 2-sum. Note that sometimes, to ensure that the 2-sum has more elements than its parts, the definition of $M \oplus_2 N$ requires that each of M and N has at least three elements.

The above definition of the 2-sum and information on representable matroids found in Section 2.3 will help us to prove the next result.

Proposition 4.12. *The 2-sum of two regular matroids is never round.*

Proof. Let M_1 and M_2 be regular matroids whose ground sets meet in a single element p. Let matroids M_1 and M_2 be represented by binary matrices A_1 and A_2, respectively, in row echelon form. Then the matrix in Figure 4.3 is a totally unimodular representation for a parallel connection $P(M_1, M_2)$ with respect to p.

$$\begin{array}{c}
\begin{array}{ccc} E(M_1)-p & p & E(M_2)-p \end{array} \\
\left[\begin{array}{c|c|c}
 & 0 & \\
 & 0 & \\
A_1 & \vdots & 0 \\
 & 0 & \\
 & 0 & \\ \cline{2-3}
 & 1 & \\ \cline{1-2}
 & 0 & \\
 & 0 & \\
0 & \vdots & A_2 \\
 & 0 & \\
 & 0 &
\end{array}\right]
\end{array}$$

Figure 4.3: Matrix representation of $P(M_1, M_2)$.

Therefore, $M_1 \oplus_2 M_2 = P(M_1, M_2)\backslash p$ can be represented by a matrix in Figure 4.4.

$$\begin{array}{c}
\begin{array}{cc} E(M_1)-p & E(M_2)-p \end{array} \\
\left[\begin{array}{c|c}
A_1 & 0 \\ \hline
0 & A_2
\end{array}\right]
\end{array}$$

Figure 4.4: Representation for $M_1 \oplus_2 M_2 = P(M, N)\backslash p$.

To show that matroid $M = M_1 \oplus_2 M_2$ is not round, we will look at its cocircuits. By definition, round matroids can not have any two disjoint cocircuits. Recall, that a cocircuit, viewed as the compliment of a hyperplane, can be found in a regular matroid by fixing a row in the corresponding matrix and looking at all the columns with non-zero entries in that row. The set of such columns corresponds to a cocircuit. Applying this technique to the representation for $M_1 \oplus_2 M_2$ shown in Figure 4.4, we can see that cocircuits which originated from matroid M_1 do not intersect cocircuits which originated from matroid M_2. Therefore, matroid $M = M_1 \oplus_2 M_2$ is not round. $\square$

In the next section we will investigate the operation of 3-sum.

4.5 The Operation of 3-sum

The operation of 3-sum of two regular matroids is analogous to the operation of 3-sum of two graph. If G_1 and G_2 are graphs, each containing a 3-cycle, then to obtain their 3-sum, one first pairs the vertices of the chosen 3-cycle of G_1 with distinct vertices of the chosen 3-cycle in G_2. The paired vertices are then identified, as are the corresponding pairs of edges. Finally, all identified edges are deleted. This process is illustrated in Figure 4.5 below.

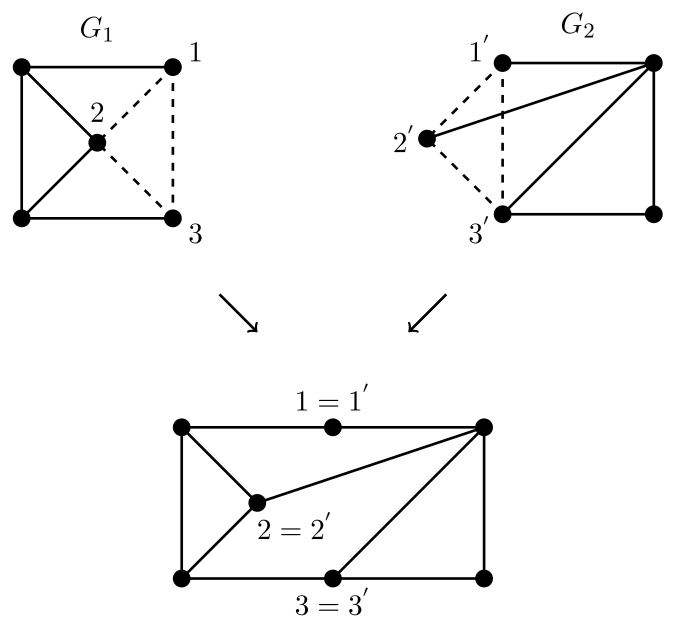


Figure 4.5: The 3-sum of G_1 and G_2.

In order to extend the operation of 3-sum from graphs to matroids, we first introduce the operation defined in Lemma 4.13 below.

Lemma 4.13. *Let M_1 and M_2 be binary matroids and $E = E(M_1) \triangle E(M_2)$. Then there is a matroid $M_1 \triangle M_2$ with ground set E whose set of circuits consists of the minimal non-empty subsets of E of the form $X_1 \triangle X_2$, where X_i is a disjoint union of circuits of M_i. Furthermore, if A is a matrix over GF(2) whose columns are indexed by the elements of $E(M_1) \cup E(M_2)$ and whose rows consist of the incidence vectors of all the circuits of M_1 and all the circuits of M_2, then*

$$M_1 \triangle M_2 = (M[A]^*) \backslash (E(M_1) \cap E(M_2)).$$

Now suppose that the ground sets of binary matroids M_1 and M_2 meet in a set T that is a triangle to both. When both $|E(M_1)|$ and $|E(M_2)|$ exceed six and neither M_1 nor M_2 has a cocircuit contained in T, we call $M_1 \triangle M_2$ the *3-sum*, $M_1 \oplus_3 M_2$, of M_1 and M_2. The next two lemmas outline key properties of 3-sums. Lemma 4.14 describes circuits in a 3-sum.

Lemma 4.14. *Let M_1 and M_2 be binary matroids such that $E(M_1) \cap E(M_2) = T$, where T is a triangle of both M_1 and M_2. Then $\mathcal{C}(M_1 \triangle M_2)$ is the union of $\mathcal{C}(M_1/T)$, $\mathcal{C}(M_2/T)$, and the collection of minimal sets of the form $C_1 \triangle C_2$ where C_i is a circuit of M_i such that $C_1 \cap T = C_2 \cap T$ and the last set has exactly one element.*

Another way to define the operation of 3-sum for binary matroids is in terms of generalized parallel connection.

Lemma 4.15. *Let M_1 and M_2 be binary matroids and $E(M_1) \cap E(M_2) = T$. Suppose that T is a 3-circuit of both M_1 and M_2, that $|E(M_1)|$ and $|E(M_2)|$ exceed six, and that T does not contain a cocircuit of M_1 and M_2. Then*

$$M_1 \oplus_3 M_2 = P_T(M_1, M_2)/T.$$

Matroid $P_T(M_1, M_2)/T$ is also called the *modular sum* of M_1 and M_2. Viewing the operation of 3-sum in terms of generalized parallel connection allows one to construct its matrix representation. The process was described and proved by a matroid theorist, Thomas Brylawski [Whi86]. It is outlined in Lemma 4.17 below.

Lemma 4.16. *Let $P_T(M_1, M_2)$ be the generalized parallel connection of $M_1(E_1\dot{\cup}T)$ and $M_2(E_2\dot{\cup}T)$. Then $P_T(M_1, M_2)$ is $\mathbb{F}$-representable if and only if there exists a representation N_1 for M_1 and N_2 for M_2 and there is a linear transformation that is nonsingular on N_1 taking the columns indexed by T in N_1 to those indexed by T in N_2.*

In this case, T has a common representation D_T in representations for M_1 and M_2, respectively, and $P_T(M_1, M_2)$ is represented by

$$N = \begin{bmatrix} D_2 & 0 & 0 \\ D_1 & D_T & D_1' \\ 0 & 0 & D_2' \end{bmatrix}$$

and

$$\begin{bmatrix} D_2 & 0 \\ D_1 & D_1' \\ 0 & D_2' \end{bmatrix} \text{ represent the associated modular sum,}$$

where

$$N_1 = \begin{bmatrix} D_2 & 0 \\ D_1 & D_T \end{bmatrix} \text{ represents } M_1,$$

$$N_2 = \begin{bmatrix} D_T & D_1' \\ 0 & D_2', \end{bmatrix} \text{ represents } M_2.$$

We will use the above result to show that the 3-sum of two regular matroids can not be round.

Proposition 4.17. *The 3-sum of two regular matroids is never round.*

Proof. Let M_1 and M_2 be regular matroids and $E(M_1) \cap E(M_2) = T$. Suppose that T is a 3-circuit of both M_1 and M_2, that $|E(M_1)|$ and $|E(M_2)|$ exceeds six, and that T does not contain a cocircuit of M_1 and M_2. Since regular matroids can be represented over $GF(2)$, then $M_1 \oplus_3 M_2 = P_T(M_1, M_2)/T$. Let matrices A_1 and A_2 in Figure 4.6 be representations of M_1 and M_2 with D_T being a representation of a 3-circuit T in both matroids.

$$A_1 = \begin{bmatrix} D_2 & 0 \\ D_1 & D_T \end{bmatrix}$$

$$A_2 = \begin{bmatrix} D_T & D_1' \\ 0 & D_2', \end{bmatrix}$$

Figure 4.6: Matrix representations of matroids M_1 and M_2.

Then by Lemma 4.16, the matrix shown in Figure 4.7 below is a representation of the 3-sum of matroids M_1 and M_2.

$$M_1 \oplus_3 M_2 = \begin{bmatrix} D_2 & 0 \\ D_1 & D_1' \\ 0 & D_2' \end{bmatrix}$$

Figure 4.7: Matrix representation of the 3-sum of matroids M_1 and M_2.

Recall, that cocircuit in a matrix corresponds to the subset of columns corresponding to non-zero entries in any fixed row. Consider one cocircuit obtained by fixing a row in submatrix D_2 and another cocircuit obtained by fixing a row in submatrix D_2'. Clearly, these cocircuits are disjoint. Therefore, the 3-sum $M_1 \oplus_3 M_2$ is not round. $\square$

Chapter 5

Conclusion

The main goal of this thesis was to study and describe regular matroids that have the additional property of being round. First, we determined the characteristics graphic round and cographic round matroids must possess. Next, we investigated the existence of other regular round matroids that are neither graphic nor cographic. Our main tool in this investigation was Seymour's Decomposition Theorem, which says that every regular matroid can be obtained by sticking together graphic matroids, cographic matroids and copies of matroid R_{10}. The operations that are used to stick these building blocks together are the direct sum, 2-sum, and 3-sum. After investigating these operations, we found that they do not produce round matroids. The results of this research are summarized in Theorem 5.1 below.

Theorem 5.1. *A regular matroid M is round if and only if $si(M)$ is either isomorphic to $M(K_n)$, $n \geq 2$, or $M^*(K_{3,3})$.*

One research direction in extending this result is to look at round matroids within a larger class of matroids. The most natural class to consider next that contains regular matroids is binary matroids. The largest binary matroids of a given rank r are called projective geometries and are denoted $PG(r-1,2)$. One can construct a matrix representation for $PG(r-1,2)$ by listing all possible non-zero column vectors of length r over the field $\mathbb{Z}_2$. Hence, $PG(r-1,2)$ contains $2^r - 1$ elements.

When investigating binary round matroids, one might consider first determining whether binary projective geometries are round. In what follows, we provide a sample of this investigation by considering the binary projective plane $PG(2,2)$. This matroid has

$2^3 - 1 = 7$ elements and its matrix representation is shown in Figure 5.1.

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right] \end{array}$$

Figure 5.1: Matrix representation of projective geometry $PG(2,2)$.

Projective geometry $PG(2,2)$ is also called the Fano plane and is denoted F_7. Its geometric representation is shown in Figure 5.2 below.

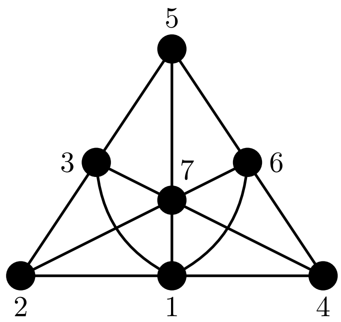


Figure 5.2: Fano Plane.

To determine whether this binary matroid is round, we are going to look at the row space of its matrix representation (Figure 5.3).

$$\begin{array}{c|c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 1 + Row 2} \\ \text{Row 1 + Row 3} \\ \text{Row 2 + Row 3} \\ \text{Row 1 + Row 2 + Row 3} \\ \text{Row 1 + Row 1} \end{array} & \left[\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Figure 5.3: Row space of a matrix representation of matroid F_7.

Recall that in a binary matroid the row space of its matrix representation is equal to the cocircuit space and cocircuits correspond to the subsets of columns with non-zero entries in any fixed row of a matrix. Therefore, by examining the row space of the matrix representation of matroid F_7 we can conclude that it has no disjoint cociruits. Thus, the binary matroid F_7 is round.

Bibliography

[CZ09] Gary Chartrand and Ping Zhang, editors. *Chromatic Graph Theory.* CRC Press, 2009.

[GM11] Gary Gordon and Jennifer McNulty. *Matroids: A Geometric Introduction.* Cambridge University Press, 2011.

[Oxl11] James Oxley. *Matroid Theory.* Oxford University Press, 2nd edition, 2011.

[Sey80] P. D. Seymour. Decomposition of regular matroids. *Journal of Combinatorial Theory*, 28:305–359, 1980.

[Whi35] Hassler Whitney. On the abstract properties of linear dependence. *American Journal of Mathematics*, 57:509–533, 1935.

[Whi86] Neil White, editor. *Encyclopedia of Mathematics and Its Applications. Theory of Matroids.* Cambridge University Press, 1986.